

YANG–MILLS CONNECTIONS ON COMPACT COMPLEX TORI

INDRANIL BISWAS

ABSTRACT. Let G be a connected reductive complex affine algebraic group and $K \subset G$ a maximal compact subgroup. Let M be a compact complex torus equipped with a flat Kähler structure and (E_G, θ) a polystable Higgs G -bundle on M . Take any C^∞ reduction of structure group $E_K \subset E_G$ to the subgroup K that solves the Yang–Mills equation for (E_G, θ) . We prove that the principal G -bundle E_G is polystable and the above reduction E_K solves the Einstein–Hermitian equation for E_G . We also prove that for a semistable (respectively, polystable) Higgs G -bundle (E_G, θ) on a compact connected Calabi–Yau manifold, the underlying principal G -bundle E_G is semistable (respectively, polystable).

1. INTRODUCTION

Let X be a compact connected Kähler manifold equipped with a Kähler form $\tilde{\omega}$. Let (E, θ) be a Higgs vector bundle on X . Given a Hermitian structure h on E , the curvature of the corresponding Chern connection on E will be denoted by \mathcal{K}_h . A Hermitian structure h is said to satisfy the Yang–Mills equation for (E, θ) if there is $c \in \mathbb{R}$ such that

$$\Lambda_{\tilde{\omega}}(\mathcal{K}_h + \theta \wedge \theta^*) = c\sqrt{-1} \cdot \text{Id}_E,$$

where $\Lambda_{\tilde{\omega}}$ is the adjoint of multiplication of forms by $\tilde{\omega}$, and θ^* is the adjoint of θ with respect to h . A Higgs bundle admits a Hermitian structure satisfying the Yang–Mills equation if and only if it is polystable [Si1], [Hi]. If $\theta = 0$, then the above Yang–Mills equation is also known as the Einstein–Hermitian equation. A holomorphic vector bundle E admits an Einstein–Hermitian metric if and only if E is polystable [UY], [Do].

More generally, let G be a connected reductive affine algebraic group over \mathbb{C} . Fix a maximal compact subgroup $K \subset G$. The center of the Lie algebra of K will be denoted by $\mathfrak{z}(\mathfrak{k})$. Let (E_G, θ) be a Higgs G -bundle on X (its definition is recalled in Section 4.1). A C^∞ reduction of structure group of E_G to K

$$E_K \subset E_G$$

is said to satisfy the Yang–Mills equation for (E, θ) if there is an element $c \in \mathfrak{z}(\mathfrak{k})$ such that

$$\Lambda_{\tilde{\omega}}(\mathcal{K} + \theta \wedge \theta^*) = c,$$

where \mathcal{K} is the curvature of the Chern connection associated to the reduction E_K and θ^* is the adjoint of θ constructed using E_K (see [At, p. 191–192, Proposition 5] for Chern connections on principal bundles). A Higgs G -bundle admits a Yang–Mills connection if

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and only if it is polystable [Si2], [BS]. As mentioned before, if $\theta = 0$, then the Yang–Mills equation is also called the Einstein–Hermitian equation.

We consider Higgs G –bundles over a torus M equipped with a flat Kähler form $\tilde{\omega}$. If (E_G, θ) is a polystable Higgs G –bundle on M , we prove that the principal G –bundle E_G is polystable. If a reduction to K

$$E_K \subset E_G$$

satisfies the Yang–Mills equation for (E, θ) , we show that E_K also satisfies the Einstein–Hermitian equation for E_G .

In the last section we observe some properties of Higgs bundles on Kähler manifolds with nonnegative tangent bundle.

2. HIGGS VECTOR BUNDLES ON A TORUS

2.1. Semistable and polystable Higgs bundles. Let M be a compact complex torus of complex dimension d . Fix a Kähler class

$$\omega \in H^{1,1}(M) \cap H^2(M, \mathbb{R}).$$

The degree of any torsionfree coherent analytic sheaf F on M is defined to be

$$\text{degree}(F) := (c_1(F) \cap \omega^{d-1}) \cap [M] \in \mathbb{R}.$$

If F is of positive rank, then

$$\mu(F) := \frac{\text{degree}(F)}{\text{rank}(F)} \in \mathbb{R}$$

is called the *slope* of F .

Let E be a holomorphic vector bundle on M . A *Higgs field* on E is a holomorphic section

$$\theta \in H^0(M, \text{End}(E) \otimes \Omega_M^1)$$

such that section

$$\theta \wedge \theta \in H^0(M, \text{End}(E) \otimes \Omega_M^2)$$

vanishes identically. A *Higgs bundle* is a holomorphic vector bundle equipped with a Higgs field.

The following lemma is well-known (see [BF], [FGN1]).

Lemma 2.1. *Let (E, θ) be a semistable Higgs bundle on M . Then the holomorphic vector bundle E is semistable.*

Proof. Assume that E is not semistable. Let

$$(2.1) \quad E_1 \subset E_2 \subset \cdots \subset E_n = E$$

be the Harder–Narasimhan filtration of E . We have

$$H^0(M, \text{End}(E_1)) = H^0(M, \text{Hom}(E_1, E))$$

because $H^0(M, \text{Hom}(E_1, E_i/E_{i-1})) = 0$ for every $i \in \{2, \dots, n\}$. Since Ω_M^1 is the trivial vector bundle of rank d , this implies that

$$\theta(E_1) \subset E_1 \otimes \Omega_M^1.$$

Therefore, E_1 contradicts the given condition that (E, θ) is semistable. Hence the vector bundle E is semistable. \square

We note that the following lemma is a consequence of Corollary 2.2 of [Bi, p. 73] (see also [FGN1]).

Lemma 2.2. *Let (E, θ) be a polystable Higgs bundle on M . Then the holomorphic vector bundle E is polystable.*

Proof. From Lemma 2.1 we know that E is semistable. Let

$$V \subset E$$

be the coherent analytic subsheaf generated by all polystable subsheaves of E with slope $\mu(E)$. This V is a polystable subsheaf with slope $\mu(E)$ [HL, page 23, Lemma 1.5.5]. We will show that

$$(2.2) \quad \theta(V) \subset V \otimes \Omega_M^1.$$

To show (2.2), fix a holomorphic trivialization of Ω_M^1 . Using this trivialization, the homomorphism θ is written as

$$\theta = (\theta_1, \dots, \theta_d),$$

where $\theta_i \in H^0(M, \text{End}(E))$ for every i . Since E is semistable, and V is polystable with $\mu(V) = \mu(E)$, it follows that $\theta_i(V)$ is polystable with $\mu(\theta_i(V)) = \mu(E)$. Therefore, (2.2) holds.

Assume that E is not polystable. So $V \neq E$. Since the Higgs bundle (E, θ) is polystable, from (2.2) and the fact that $\mu(V) = \mu(E)$ we conclude that there is a coherent analytic subsheaf

$$V' \subset E$$

such that $\mu(V') = \mu(E)$ and $V \cap V' = 0$. Let $V'' \subset V'$ be a polystable subsheaf such that $\mu(V'') = \mu(V')$. From the definition of V it follows that $V'' \subset V$. But this contradicts the condition that $V \cap V' \subset V \cap V' = 0$. So E is polystable. \square

2.2. Higgs fields on a polystable vector bundle. Let $E \rightarrow M$ be a polystable vector bundle. Our aim in this subsection is to describe all Higgs fields θ on E such that the Higgs bundle (E, θ) is polystable.

Since E is polystable, we can write

$$(2.3) \quad E = \bigoplus_{j=1}^{\ell} E_j \otimes \mathbb{C}^{n_j},$$

where

- each E_j is a stable vector bundle with $\mu(E_j) = \mu(E)$,
- E_j is not isomorphic to $E_{j'}$ if $j \neq j'$, and
- $n_j > 0$ for every j .

From the first two conditions it follows immediately that $H^0(M, \text{Hom}(E_j, E_{j'})) = 0$ if $j \neq j'$. Therefore, we have

$$(2.4) \quad H^0(M, \text{Hom}(E_j, E_{j'}) \otimes \Omega_M^1) = 0 \quad \text{if } j \neq j'.$$

Since E_j is stable, we also have

$$(2.5) \quad H^0(M, \text{End}(E_j)) = \mathbb{C}.$$

In view of (2.4) and (2.5), any $\beta \in H^0(M, \text{End}(E))$ can be written as

$$\beta = \bigoplus_{j=1}^{\ell} \text{Id}_{E_j} \otimes T_j$$

in terms of the isomorphism in (2.3), where

$$T_j \in M(n_j, \mathbb{C}) = \text{End}_{\mathbb{C}}(\mathbb{C}^{n_j}).$$

As before, fix a holomorphic trivialization of Ω_M^1 . Using this trivialization, any $\theta \in H^0(M, \text{End}(E) \otimes \Omega_M^1)$ can be written as

$$\theta = (\theta_1, \dots, \theta_d),$$

where $\theta_i \in H^0(M, \text{End}(E))$.

Take any

$$(2.6) \quad \theta \in H^0(M, \text{End}(E) \otimes \Omega_M^1).$$

Write

$$\theta = (\theta_1, \dots, \theta_d),$$

as above. Let

$$(2.7) \quad \theta_i = \bigoplus_{j=1}^{\ell} \text{Id}_{E_j} \otimes T_j^i,$$

where $T_j^i \in M(n_j, \mathbb{C})$.

Proposition 2.3. *The pair (E, θ) in (2.6) is a polystable Higgs bundle if and only if*

- (1) $T_j^i T_j^k = T_j^k T_j^i$ (see (2.7)) for all $i, k \in \{1, \dots, d\}$ and all j , and
- (2) each T_j^i is semisimple.

Proof. First assume that the two conditions in the proposition are satisfied. The first condition implies that $\theta \wedge \theta = 0$. The second condition implies that (E, θ) can be expressed as a direct sum of stable Higgs bundles of same slope. Therefore, (E, θ) is polystable.

Now assume that (E, θ) is a polystable Higgs bundle. Since $\theta \wedge \theta = 0$, the first condition in the proposition holds. The Higgs bundle (E, θ) is a direct sum of stable Higgs bundles of same slope. From this it follows that the second condition in the proposition is satisfied. \square

Remark 2.4. A sum of commuting semisimple matrices is again semisimple. Therefore, the two conditions in Proposition 2.3 are independent of the choice of the trivialization of Ω_M^1 .

3. YANG–MILLS HERMITIAN METRIC ON POLYSTABLE HIGGS BUNDLES

Let $\text{Aut}^0(M)$ denote the connected component, containing the identity element, of the group of holomorphic automorphisms of M . The complex manifold $\text{Aut}^0(M)$ is isomorphic to M . If we consider M as a complex abelian Lie group, then $\text{Aut}^0(M)$ coincides with the group of translations of M .

There is a unique Kähler form $\tilde{\omega}$ on M such that

- the cohomology class of $\tilde{\omega}$ coincides with ω , and
- the form $\tilde{\omega}$ is preserved by the action of $\text{Aut}^0(M)$ on M .

The Kähler structure on M given by $\tilde{\omega}$ is flat. Fix the Kähler form $\tilde{\omega}$ on M .

Proposition 3.1. *Let (E, θ) be a polystable Higgs bundle on M . There is a Yang–Mills Hermitian metric h on E for the Higgs field θ such that h satisfies the Einstein–Hermitian equation for the polystable vector bundle E .*

Proof. Fix a trivialization of Ω_M^1 using holomorphic sections of Ω_M^1 that are pointwise orthonormal. Such a trivialization exists because the connection on Ω_M^1 corresponding to $\tilde{\omega}$ is flat with trivial monodromy. Take the endomorphisms T_j^i in Proposition 2.3. Take any $j \in \{1, \dots, \ell\}$. Since $T_j^i T_j^k = T_j^k T_j^i$ for all $i, k \in \{1, \dots, d\}$, we have a simultaneous eigenspace decomposition of \mathbb{C}^{n_j} for the eigenvalues of T_j^i , $i \in \{1, \dots, d\}$. Fix an inner product h_j on \mathbb{C}^{n_j} such that the above decomposition of \mathbb{C}^{n_j} given by the eigenspaces of $\{T_j^i\}_{i=1}^d$ is orthogonal.

Fix an Einstein–Hermitian structure h'_j on the stable vector bundle E_j in (2.3). The Hermitian structures h_j and h'_j together produce a Hermitian structure on the vector bundle $E_j \otimes \mathbb{C}^{n_j}$ in (2.3). These together in turn define a Hermitian structure h on E using the isomorphism in (2.3) after imposing the condition that the subbundles $E_j \otimes \mathbb{C}^{n_j}$ in (2.3) are orthogonal.

The above Hermitian structure h on E clearly satisfies the Einstein–Hermitian equation for the polystable vector bundle E .

Let $\theta^* \in C^\infty(M; \text{End}(E) \otimes \Omega_M^{0,1})$ be the adjoint of θ . From the construction of h it follows that $\theta \wedge \theta^* = 0$. Using this it is straightforward to check that h satisfies the Yang–Mills equation for the Higgs bundle (E, θ) . \square

Theorem 3.2. *Let (E, θ) be a polystable Higgs bundle on M . Let h' be a Yang–Mills Hermitian metric on E for the Higgs field θ . Then h' satisfies the Einstein–Hermitian equation for the polystable vector bundle E .*

Proof. Consider the Yang–Mills Hermitian metric h on E constructed in Proposition 3.1. The two Hermitian structures h and h' differ by a holomorphic automorphism of E . In other words, there is a holomorphic automorphism

$$T : E \longrightarrow E$$

such that

$$(3.1) \quad h'(v, w) = h(T(v), T(w))$$

for all $v, w \in E_x$ and all $x \in M$. From this we will derive that h' satisfies the Einstein–Hermitian equation for the polystable vector bundle E .

Consider the holomorphic vector bundle $End(E) = E \otimes E^*$. The Hermitian structure h on E produces a Hermitian structure on $End(E)$. The corresponding Chern connection ∇ on $End(E)$ is Einstein–Hermitian, because h satisfies the Einstein–Hermitian equation. Note that $c_1(End(E)) = 0$. Therefore, the mean curvature of the Einstein–Hermitian connection ∇ on $End(E)$ vanishes identically (see [Ko, p. 51] for mean curvature). Therefore, any holomorphic section of $End(E)$ is flat with respect to ∇ [Ko, p. 52, Theorem 1.9]. In particular, the section T in (3.1) is flat with respect to ∇ .

Since h satisfies the Einstein–Hermitian equation for E , and T is flat with respect to the connection ∇ given by h , it follows that h' defined by (3.1) also satisfies the Einstein–Hermitian equation for E . \square

4. HIGGS G –BUNDLES ON M

4.1. Semistable and polystable Higgs G –bundles. Let G be a connected reductive affine algebraic group defined over \mathbb{C} . The Lie algebra of G will be denoted by \mathfrak{g} . For a holomorphic principal G –bundle E_G on M , let $\text{ad}(E_G) := E_G \times^G \mathfrak{g}$ be the adjoint bundle. A section

$$\theta \in H^0(M, \text{ad}(E_G) \otimes \Omega_M^1)$$

is called a *Higgs field* on E_G if $\theta \wedge \theta = 0$. A *Higgs G –bundle* is a holomorphic principal G –bundle equipped with a Higgs field.

The proof of the following lemma is very similar to the proof of Lemma 2.1.

Lemma 4.1. *Let (E_G, θ) be a semistable Higgs G –bundle on M . Then the principal G –bundle E_G is semistable.*

Proof. Assume that the principal G –bundle E_G is not semistable. Let

$$E_P \subset E_G$$

be the Harder–Narasimhan reduction of E_G over the dense open subset U associated to E_G . We have

$$H^0(U, \text{ad}(E_G)/\text{ad}(E_P)) = 0$$

[AAB, p. 705, Corollary 1]. Since the vector bundle Ω_M^1 is trivial, this implies that the image of θ in $H^0(U, (\text{ad}(E_G)/\text{ad}(E_P)) \otimes \Omega_M^1)$ vanishes identically. In other words,

$$\theta \in H^0(U, \text{ad}(E_P) \otimes \Omega_M^1).$$

Therefore, the above reduction E_P contradicts the given condition that the Higgs G –bundle (E_G, θ) is semistable. Consequently, the principal G –bundle E_G is semistable. \square

Lemma 4.1 and Lemma 4.2 are proved in [FGN2] under the assumption that M is an elliptic curve.

Lemma 4.2. *Let (E_G, θ) be a polystable Higgs G –bundle on M . Then the principal G –bundle E_G is polystable.*

Proof. Since (E_G, θ) is polystable, it admits a Yang–Mills connection ∇ [BS, p. 554, Theorem 4.6]. Let $\text{ad}(\theta)$ be the Higgs field on the vector bundle $\text{ad}(E_G)$ induced by θ . The connection on $\text{ad}(E_G)$ induced by ∇ satisfies Yang–Mills equation for the Higgs bundle $(\text{ad}(E_G), \text{ad}(\theta))$. Therefore, $(\text{ad}(E_G), \text{ad}(\theta))$ is polystable. Hence the vector bundle

$\mathrm{ad}(E_G)$ is polystable by Lemma 2.2. This implies that the principal G -bundle E_G is polystable [AB, p. 224, Corollary 3.8]. \square

4.2. A Levi reduction associated to a semisimple section. Let E_G be a holomorphic principal G -bundle over M . Let

$$(4.1) \quad \eta \in H^0(M, \mathrm{ad}(E_G))$$

be a section such that $\eta(x) \in \mathrm{ad}(E_G)_x$ is semisimple for every $x \in M$. Since the Lie algebra $\mathrm{ad}(E_G)_x$ is identified with the Lie algebra \mathfrak{g} of G up to an inner automorphism, the element $\eta(x) \in \mathrm{ad}(E_G)_x$ defines a conjugacy class in \mathfrak{g} . Let

$$C_x \subset \mathfrak{g}$$

denote this orbit of G in \mathfrak{g} given by $\eta(x)$.

Proposition 4.3. *The above conjugacy class C_x is independent of the point x .*

Proof. Fix a maximal torus $T \subset G$. Let $W := N(T)/T$ be the corresponding Weyl group, where $N(T) \subset G$ is the normalizer of T . The Lie algebra of T will be denoted by \mathfrak{t} . The space of semisimple conjugacy classes in \mathfrak{g} is identified with the quotient \mathfrak{t}/W .

Since \mathfrak{t}/W is an affine variety, and M is a compact connected complex manifold, there are no nonconstant holomorphic maps from M to \mathfrak{t}/W . This immediately implies that C_x is independent of the point x . \square

Fix an element

$$(4.2) \quad \eta' \in C_x \subset \mathfrak{g}.$$

Let

$$(4.3) \quad L = C(\eta') \subset G$$

be the centralizer of η' . It is known that L is a Levi subgroup of G [DM, p. 26, Proposition 1.22]; we recall that a Levi subgroup of G is a maximal connected reductive subgroup of some parabolic subgroup of G .

Proposition 4.4. *Given η and η' as above, the principal G -bundle E_G has a natural holomorphic reduction of structure group to the subgroup L defined in (4.3).*

Proof. For any $g \in G$, let

$$\mathrm{Ad}(g) : \mathfrak{g} \longrightarrow \mathfrak{g}$$

be the Lie algebra automorphism corresponding to the automorphism of the group G defined by $z \longmapsto g^{-1}zg$. We recall that $\mathrm{ad}(E_G)$ is the quotient of $E_G \times \mathfrak{g}$ where two point $(y_1, v_1), (y_2, v_2) \in E_G \times \mathfrak{g}$ are identified if there is an element $g \in G$ such $y_2 = y_1g$ and $v_2 = \mathrm{Ad}(g)(v_1)$. Let

$$q : E_G \times \mathfrak{g} \longrightarrow \mathrm{ad}(E_G)$$

be the quotient map.

Let

$$p_1 : E_G \times \mathfrak{g} \longrightarrow E_G$$

be the projection to the first factor. Define

$$(4.4) \quad \mathcal{Z} := p_1((q^{-1}(\eta(M))) \cap (E_G \times \eta')) \subset E_G.$$

It is straightforward to check that \mathcal{Z} is a holomorphic reduction of structure group of E_G to the subgroup L . \square

If η' in (4.2) is replaced by $\text{Ad}(g)(\eta')$ for some $g \in G$, then the subgroup L in (4.3) gets replaced by $g^{-1}Lg$.

Corollary 4.5. *If η' in (4.2) is replaced by $\text{Ad}(g)(\eta')$ for some $g \in G$, then \mathcal{Z} in (4.4) gets replaced by $\mathcal{Z}g \subset E_G$.*

Proof. This follows immediately from the construction in (4.4). \square

Let

$$E_L \subset E_G$$

be the reduction of structure group to L constructed in Proposition 4.4. Let $\text{ad}(E_L)$ be the adjoint vector bundle for E_L .

Corollary 4.6. *The subbundle $\text{ad}(E_L) \subset \text{ad}(E_G)$ is independent of the choice of the element η' in (4.2).*

Proof. This follows immediately from Corollary 4.5. \square

Corollary 4.7. *For any $x \in M$, the subalgebra $\text{ad}(E_L)_x \subset \text{ad}(E_G)_x$ coincides with the centralizer of $\eta(x) \in \text{ad}(E_G)_x$.*

Proof. This follows from the construction of E_L in (4.4). \square

4.3. A Levi reduction associated to a semisimple Higgs field. Let (E_G, θ) be a Higgs G -bundle on M . For any $\alpha \in H^0(M, TM)$, we have

$$\theta(\alpha) \in H^0(M, \text{ad}(E_G)).$$

Take a basis $\{\alpha_1, \dots, \alpha_d\}$ of $H^0(M, TM)$.

Lemma 4.8. *If $\theta(\alpha_i)$ is pointwise semisimple for every $i \in \{1, \dots, d\}$, then for any $\alpha \in H^0(M, TM)$, the section $\theta(\alpha)$ is pointwise semisimple.*

Proof. Since θ is a Higgs field, we have $[\theta(\alpha_i), \theta(\alpha_j)] = 0$ for every $i, j \in \{1, \dots, d\}$. The lemma follows from the fact that a sum of commuting semisimple elements of \mathfrak{g} is again semisimple. \square

Assume that $\theta(\alpha_i)$ is pointwise semisimple for every $i \in \{1, \dots, d\}$. Let $L_1 \subset G$ be the Levi subgroup constructed as in (4.3) for the section $\theta(\alpha_1)$. Let

$$E_{L_1} \subset E_G$$

be the reduction constructed as in Proposition 4.3 for $\theta(\alpha_1)$. Since $[\theta(\alpha_1), \theta(\alpha_2)] = 0$, from Corollary 4.7 we know that

$$\theta(\alpha_2) \subset H^0(M, \text{ad}(E_{L_1})) \subset H^0(M, \text{ad}(E_G)).$$

Therefore, proceeding inductively, we get from the Higgs field θ

- a Levi subgroup $L \subset G$, and

- a holomorphic reduction of structure group

$$(4.5) \quad E_L \subset E_G$$

to L .

The subgroup L is unique up to a conjugation. If L is replaced by $g^{-1}Lg$ for some $g \in G$, then E_L gets replaced by $E_L g$. Consequently, the subbundle

$$\mathrm{ad}(E_L) \subset \mathrm{ad}(E_G)$$

is uniquely determined by θ . Also, note that

$$(4.6) \quad \theta \in H^0(M, \mathrm{ad}(E_L) \otimes \Omega_M^1) \subset H^0(M, \mathrm{ad}(E_G) \otimes \Omega_M^1).$$

From Corollary 4.7 it follows that for every $i \in \{1, \dots, d\}$ and $x \in M$, the subalgebra $\mathrm{ad}(E_L)_x \subset \mathrm{ad}(E_G)_x$ is contained in the centralizer of $\theta(\alpha_i)(x)$. More precisely, $\mathrm{ad}(E_L)_x$ is the centralizer of the subset $\{\theta(\alpha_1)(x), \dots, \theta(\alpha_d)(x)\} \subset \mathrm{ad}(E_G)_x$.

5. YANG–MILLS STRUCTURE ON POLYSTABLE HIGGS G –BUNDLES ON M

Let (E_G, θ) be a polystable Higgs G –bundle on M .

Proposition 5.1. *For any $i \in \{1, \dots, d\}$ and $x \in M$, the element $\theta(\alpha_i)(x) \in \mathrm{ad}(E_G)_x$ is semisimple.*

Proof. Let

$$(5.1) \quad Z(G) \subset G$$

be the connected component of the center of G containing the identity element. Take a finite dimensional holomorphic representation

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

such that $\rho(Z(G))$ is contained in the center of $\mathrm{GL}(V)$. Let

$$E_V := E_G \times^G V \longrightarrow M$$

be the vector bundle associated to E_G for this G –module V . The Higgs field θ induces a Higgs field on E_V . This induced Higgs field on E_V will be denoted by θ_V . The connection on E_V induced by a Yang–Mills connection for (E_G, θ) satisfies the Yang–Mills equation for the Higgs bundle (E_V, θ_V) . This implies that (E_V, θ_V) is polystable.

Now from Proposition 2.3 we conclude that $\theta_V(\alpha_i)(x) \in \mathrm{End}(E_V)_x$ is semisimple for every $i \in \{1, \dots, d\}$ and $x \in M$. Since ρ is an arbitrary holomorphic representation such that $\rho(Z(G))$ is contained in the center of $\mathrm{GL}(V)$, this implies that $\theta(\alpha_i)(x)$ is semisimple. \square

Consider the Higgs L –bundle (E_L, θ) constructed from the given polystable Higgs G –bundle (E_G, θ) (see (4.5), (4.6)). We note that (E_L, θ) is polystable because (E_G, θ) is so. From Lemma 4.2 we know that the principal L –bundle E_L is polystable.

As in Section 3, fix the Kähler form $\tilde{\omega}$ on M . Fix a maximal compact subgroup

$$K_L \subset L.$$

Let

$$E_{K_L} \subset E_L$$

be a C^∞ reduction of structure group to K_L that solves the Yang–Mills equation for (E_L, θ) (see [BS, p. 554, Theorem 4.6]).

Proposition 5.2. *The above reduction*

$$E_{K_L} \subset E_L$$

solves the Einstein–Hermitian equation for the polystable principal L -bundle E_L .

Proof. We observed earlier that for every $i \in \{1, \dots, d\}$ and $x \in M$, the subalgebra $\text{ad}(E_L)_x \subset \text{ad}(E_G)_x$ is contained in the centralizer of $\theta(\alpha_i)(x)$. Therefore, for every $i \in \{1, \dots, d\}$ and $x \in M$, the element $\theta^*(\overline{\alpha_i})(x) \in \text{ad}(E_L)_x$ also is contained in the center of $\text{ad}(E_L)_x$. Consequently, we have $\theta \wedge \theta^* = 0$. This immediately implies that the reduction

$$E_{K_L} \subset E_L$$

solves the Einstein–Hermitian equation for the polystable principal L -bundle E_L . \square

Fix a maximal compact subgroup

$$K \subset G$$

such that $K \cap L = K_L$.

Theorem 5.3. *Let (E_G, θ) be a polystable Higgs G -bundle on M . Let*

$$E_K \subset E_G$$

be a C^∞ reduction of structure group to K that solves the Yang–Mills equation for (E_G, θ) . Then the reduction $E_K \subset E_G$ solves the Einstein–Hermitian equation for the polystable principal G -bundle E_G .

Proof. As before, let (E_L, θ) be the Higgs L -bundle constructed from the polystable Higgs G -bundle (E_G, θ) (see (4.5), (4.6)). Take a C^∞ reduction

$$E_{K_L} \subset E_L$$

that solves the Yang–Mills equation for (E_L, θ) . Let

$$E'_K := E_{K_L}(K) \longrightarrow M$$

be the principal K -bundle obtained by extending the structure group of E_{K_L} using the inclusion of K_L in K . We note that E'_K is a reduction of structure group of E_G to K because E_{K_L} is a reduction of structure group of E_G to K_L . The above reduction

$$E'_K \subset E_G$$

solves the Yang–Mills equation for (E_G, θ) because the reduction $E_{K_L} \subset E_L$ solves the Yang–Mills equation for (E_L, θ) .

Therefore, there is a holomorphic automorphism T of E_G such that $E_K = T(E'_K)$.

Let $\text{Ad}(E_G) = E_G \times^G G \longrightarrow M$ be the holomorphic fiber bundle associated to E_G for the adjoint action of G on itself. It can be shown that the holomorphic sections of $\text{Ad}(E_G)$ are flat with respect to the connection on $\text{Ad}(E_G)$ induced by the connection on E_G given by the reduction E'_K . To prove this, take any finite dimensional holomorphic G -module

$$\rho : G \longrightarrow \text{GL}(V)$$

such that $\rho(Z(G))$ (see (5.1)) is contained in the center of $\mathrm{GL}(V)$. Let

$$E_V := E_G \times^G V \longrightarrow M$$

be the associated vector bundle. The Einstein–Hermitian connection on E_G given by the reduction E'_K produces an Einstein–Hermitian connection on $\mathrm{End}(E_V)$; this Einstein–Hermitian connection on $\mathrm{End}(E_V)$ will be denoted by ∇' .

Given any holomorphic section T' of $\mathrm{Ad}(E_G)$, let T'' be the automorphism of E_V given by T' . As done in the proof of Theorem 3.2, using [Ko, p. 52, Theorem 1.9] we conclude that the section T'' of $\mathrm{End}(E)$ is flat with respect to ∇' . From this it follows that the automorphism T' of E_G is flat with respect to the connection on $\mathrm{Ad}(E_G)$ induced by the connection on E_G given by the reduction E'_K .

In particular, the earlier automorphism T is flat with respect to the connection on $\mathrm{Ad}(E_G)$ corresponding to the reduction E'_K . From this it follows that the reduction $E_K \subset E_G$ solves the Einstein–Hermitian equation for the polystable principal G -bundle E_G . \square

6. HIGGS BUNDLES ON KÄHLER MANIFOLDS WITH NONNEGATIVE TANGENT BUNDLE

Let X be a compact connected Kähler manifold equipped with a Kähler class ω . Let

$$W_1 \subset \cdots \subset W_m = \Omega_X^1$$

be the Harder–Narasimhan filtration of Ω_X^1 .

Lemma 6.1. *Assume that $\mu_{\max}(\Omega_X^1) := \mu(W_1) < 0$. Let (E, θ) be a semistable Higgs bundle on X . Then $\theta = 0$.*

Proof. To prove that E is semistable, consider E_1 in (2.1). We have

$$H^0(X, \mathrm{Hom}(E_1, (E/E_1) \otimes \Omega_X^1)) = 0$$

because $\mu_{\max}((E/E_1) \otimes \Omega_X^1) = \mu_{\max}(E/E_1) + \mu_{\max}(\Omega_X^1) < \mu(E_1) + 0 = \mu(E_1)$. From this it follows that $\theta(E_1) \subset E_1 \otimes \Omega_X^1$. Since (E, θ) is semistable, this implies that $E = E_1$. So E is semistable.

Since E is semistable,

$$\mu_{\max}(E \otimes \Omega_X^1) = \mu(E) + \mu_{\max}(\Omega_X^1) < \mu(E).$$

Hence $H^0(X, \mathrm{End}(E) \otimes \Omega_X^1) = 0$. In particular, $\theta = 0$. \square

Combining the proofs of Lemma 4.1 and Lemma 6.1 it is easy to deduce that Lemma 6.1 remains valid for Higgs G -bundles on X . The only point to note is that [AAB, p. 705, Corollary 1] (which is used in the proof of Lemma 4.1) is proved by showing that $\mu_{\max}(\mathrm{ad}(E_G)/\mathrm{ad}(E_P)) < 0$.

6.1. Higgs bundles on Calabi–Yau manifolds. Let X be a compact connected Kähler manifold such that $c_1(TX) \in H^2(X, \mathbb{Q})$ is zero. These are known as Calabi–Yau manifolds. Fix a Kähler class ω on X . A celebrated theorem of Yau says that there is a Kähler form $\tilde{\omega}$ in the class ω such that the Ricci curvature for $\tilde{\omega}$ vanishes identically [Ya] (this was conjectured earlier by Calabi). In particular, $\tilde{\omega}$ is an Einstein–Hermitian structure on Ω_X^1 . This implies that the vector bundle Ω_X^1 is polystable.

Lemma 6.2. *Let (E, θ) be a semistable Higgs bundle on X . Then the vector bundle E is semistable. This is also true for Higgs G -bundles, meaning if (E_G, θ) is a semistable Higgs G -bundle on X , then the underlying principal G -bundle E_G is semistable.*

Proof. Since Ω_X^1 is polystable of slope zero, the proof of it given in Lemma 6.1 remains valid. To prove for Higgs G -bundles, just note that for $\text{ad}(E_P)$ in the proof of Lemma 4.1 we have $\mu_{\max}(\text{ad}(E_G)/\text{ad}(E_P)) < 0$ (see the proof of Corollary 1 in [AAB, p. 705]). \square

Lemma 6.3. *Let (E, θ) be a polystable Higgs bundle on X . Then the vector bundle E is polystable.*

Proof. From Lemma 6.2 we know that E is semistable. As in the proof of Lemma 2.2,

$$V \subset E$$

is the coherent analytic subsheaf generated by all polystable subsheaves of E with slope $\mu(E)$. Let $\theta' : TX \otimes E \rightarrow E$ be the following composition homomorphism

$$TX \otimes E \xrightarrow{\text{Id}_{TX} \otimes \theta} TX \otimes \Omega_X^1 \otimes E \xrightarrow{\text{trace} \otimes \text{Id}_E} E.$$

Since both TX and V are polystable, it follows that $TX \otimes V$ is polystable. Also, note that $\mu(TX \otimes V) = \mu(V) = \mu(E)$. Therefore, the image

$$\theta'(TX \otimes V) \subset E$$

is polystable with $\mu(\theta'(TX \otimes V)) = \mu(E)$. Hence, we have

$$\theta'(TX \otimes V) \subset V.$$

This implies that $\theta(V) \subset V \otimes \Omega_X^1$. Now the last part of the proof of Lemma 2.2 shows that E is polystable. \square

Lemma 6.4. *Let (E_G, θ) be a polystable Higgs G -bundle on X . Then the principal G -bundle E_G is polystable.*

Proof. The proof is identical to the proof of Lemma 4.2. \square

REFERENCES

- [AAB] B. Anchouche, H. Azad and I. Biswas, Harder–Narasimhan reduction for principal bundles over a compact Kähler manifold, *Math. Ann.* **323** (2002), 693–712.
- [AB] B. Anchouche and I. Biswas, Einstein–Hermitian connections on polystable principal bundles over a compact Kähler manifold, *Amer. Jour. Math.* **123** (2001), 207–228.
- [At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.
- [BF] I. Biswas and C. Florentino, Commuting elements in reductive groups and Higgs bundles on abelian varieties, *Jour. Alg.* **388** (2013), 194–202.
- [BS] I. Biswas and G. Schumacher, Yang–Mills equation for stable Higgs sheaves, *Inter. Jour. Math.* **20** (2009), 541–556.
- [Bi] I. Biswas, Stable Higgs bundles on compact Gauduchon manifolds, *Com. Ren. Math. Acad. Sci. Paris* **349** (2011), 71–74.
- [DM] F. Digne and J. Michel, *Representations of finite groups of Lie type*, London Math. Soc. Stud. Texts 21, Cambridge University Press, Cambridge 1991.
- [Do] S. K. Donaldson, Anti self–dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. London Math. Soc.* **50** (1985), 1–26.

- [FGN1] E. Franco, O. García-Prada and P. E. Newstead, Higgs bundles over elliptic curves, *Ill. Jour. Math.* (in press).
- [FGN2] E. Franco, O. García-Prada and P. E. Newstead, G –Higgs bundles over elliptic curves, preprint, arXiv:1310.2168.
- [Hi] N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* **55** (1987), 59–126.
- [HL] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.
- [Ko] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, 15. Kanô Memorial Lectures, 5. Princeton University Press, Princeton, NJ, 1987.
- [Si1] C. T. Simpson, Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization, *Jour. Amer. Math. Soc.* **1** (1988), 867–918.
- [Si2] C. T. Simpson, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5–95.
- [UY] K. Uhlenbeck and S.-T. Yau, On the existence of Hermitian–Yang–Mills connections in stable vector bundles, *Commun. Pure Appl. Math.* **39** (1986), 257–293.
- [Ya] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, BOMBAY 400005, INDIA

E-mail address: `indranil@math.tifr.res.in`